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Applications of Hook Young Diagrams to P.I. Algebras

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(a) The multiplicities m_λ in the cocharacters $\chi_n(A)$ of (any P.I. algebra A) are polynomially bounded. (b) A hook containing $\chi_n(A \otimes B)$ is obtained from the hooks containing $\chi_n(A)$ and $\chi_n(B)$. These results are obtained by applying a theory of hook Young diagrams to P.I. algebras, and they generalize results known for algebras satisfying Capelli identities.

0. INTRODUCTION

Let $\chi_n(A) = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda[\lambda]$ be the cocharacter of a P.I. algebra in characteristic zero, and let $\mathrm{Par}(n) \supseteq H(k, l; n) = \{(\lambda_1, \lambda_2, ...) \in \mathrm{Par}(n) \mid \lambda_j \leqslant l$ for $j \geqslant k+1\}$. Amitsur and Regev [1], proved that there exist $k, l \geqslant 0$ such that $m_\lambda = 0$ if $\lambda \notin H(k, l; n)$, $\chi_n(A) = \sum_{\lambda \in H(k,l;n)} m_\lambda[\lambda]$. A priori, the m_λ's are only exponentially bounded, $m_\lambda \leqslant d_\lambda$. The main result in this paper is that the m_λ's are polynomially bounded. This was proved in [2] under the assumption that A satisfies a Capelli identity. Many P.I. algebras do not satisfy a Capelli identity: If A is any non-nilpotent P.I. algebra, then $A \otimes E$ does not satisfy any Capelli identity, where E is the infinite Grassmann algebra. It is interesting to note that algebras of the form $A \otimes E$ enter the proof of the general case in a crucial way. The above special case was proved in [2] using $gl(V)$ (or $GL(V)$, the general linear Lie algebra—or group) and S_n representations. Here we obtain the proof of the general case by applying $pl(V)$ (the general linear Lie superalgebra) and S_n representations. This is done in Sections 2 and 3. The tools for handling hook shaped Young diagrams and $pl(V)$ representations were developed in [3]. We assume the reader is familiar with these results.

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In Section 1 we bring an immediate application of that "hook theory" to P.I. algebras: Given two P.I. algebras A and B we construct a hook containing the cocharacter $\chi_n(A \otimes B)$ in terms of the corresponding hook for $\chi_n(A)$, $\chi_n(B)$. In particular, this yields a way of constructing explicit identities for $A \otimes B$.

1

We begin by quoting [1, Theorem C]. Recall that $\mathrm{Par}(n) \supseteq H(k, l; n) = \{\lambda(\lambda_1, \lambda_2, \ldots) \mid \lambda_{k+1}, \lambda_{k+2}, \ldots \leqslant l\}$.

THEOREM 1 [1, Theorem C]. *Let A satisfy an identity of degree d, let $k, l > e(d-1)^4$ $(e = 2.7\ldots)$, and let $\chi_n(A)$, $n = 1, 2, \ldots$, be the cocharacter sequence of A. Then, for all n, $\chi_n(A) = \sum_{\lambda \in H(k,l;n)} m_\lambda[\lambda]$.*

For such (k, l) we write $\chi(A) \subseteq H(k, l)$. Obviously, there might be other hooks contaiing the cocharacters $\chi_n(A)$. We thus introduce

Notation 2. $H(A) = \{(k, l) \mid \chi_n(A) = \sum_{\lambda \in H(k,l;n)} m_\lambda[\lambda]\} = \{(k, l) \mid \chi(A) \subseteq H(k, l)\}$.

Given a P.I. algebra, a basic problem is to find $(k, l) \in H(A)$. We do this for $A \otimes B$ in

THEOREM 3. *If $\chi(A) \subseteq H(k_1, l_1)$ and $\chi(B) \subseteq H(k_2, l_2)$, then $\chi(A \otimes B) \subseteq H(k, l)$, where $k = k_1 k_2 + l_1 l_2$, $l = k_1 l_2 + k_2 l_1$.*

Proof. By [10], $\chi_n(A \otimes B) \leqslant \chi_n(A) \otimes \chi_n(B)$. By [3, Sect. 3], if $\lambda \in H(k_1, l_1; n)$, $\mu \in H(k_2, l_2; n)$, then $[\lambda] \otimes [\mu] \subseteq H(k, l)$, i.e., all the irreducible components of $[\lambda] \otimes [\mu]$ are from $H(k, l; n)$. The proof now follows. Q.E.D.

Using the notations and results of [1], Theorem 3 takes the following equivalent form:

THEOREM 3′. *Let A satisfy $E_\lambda^*[x; y]$ and let B satisfy $E_\mu^*[x,; y]$, where $\lambda = (l_1^{k_1})$, $\mu = (l_2^{k_2})$, then $A \otimes B$ satisfies $E_\nu^*[x; y]$, where $\nu = (l^k)$, $k = k_1 k_2 + l_1 l_2$, $l = k_1 l_2 + k_2 l_1$. This generalizes* [10, *Theorem* 13].

Let $\chi(A) \subseteq H(k, l)$. Recall that if $\lambda \notin H(k, l; n)$ and $I_\lambda \subset FS_n$ is the corresponding two-sided ideal in FS_n, then every element (polynomial) in I_λ is an identity for A [8]. So we may choose the $(k+1) \times (l+1)$ rectangle to conslruct explicit identities—of degree $(k+1)(l+1)$—for A. Thus, Theorem 3 yields a method, which we name the "hook method," for constructing explicit identities for $A \otimes B$. A second method, which we call the "codimension method" is given in [8, Theorem 4.4]. We do as an example $M_k(F) \otimes E = M_k(E)$. Since $\chi(E) \subseteq H(1,1)$ [7] and $\chi(F_k) \subseteq H(k^2, 0)$ [9], hence $\chi(F_k \otimes E) \subseteq H(k^2, k^2)$; so the hook method gives identities for $M_k(E)$ of degree $(k^2+1)^2$, for example, $(s_{k^2+1}[x])^{k^2+1}$. The codimension method yields only $(s_{k^2+1}[x])^t$, where $t > 4k^8$.

2

Let $F\langle x\rangle = F\langle x_1, x_2, \ldots\rangle$ be the free associative, noncommutative algebra in countably many variables, Let A be a P.I. algebra, $I(A) = Q \subseteq F\langle x\rangle$ its T-ideal of identities, $V_n =$ the multilinear polynomials of degree n in $x_1, x_2, \ldots, x_n$, $Q_n = Q \cap V_n$. As in [8], $V_n \equiv FS_n$ and Q_n becomes a left ideal. Finally let V be a vector space, $\dim V = k$, and $T(V) = \sum_{n=0}^{\infty} \oplus V^{\otimes n}$ its tensor algebra. Thus $T(V) \equiv F\langle x_1, \ldots, x_k\rangle$ and $V^{\otimes n} \cap Q$ is defined. The symmetric group S_n acts from the right (classically) on $V^{\otimes n}$ by permuting coordinates, so $V^{\otimes n} \cdot Q_n$ is also defined. A standard multilinearization argument proves

LEMMA 4. $V^{\otimes n} \cap Q = V^{\otimes n} \cdot Q_n$.

DEFINITION. $G(A) = F\langle x\rangle / Q$ is the universal algebra for A in infinity many variables. Similarly, $G(A; k) = T(V)/Q \cap T(V)$ is the universal algebra for A in k variables. By the lemma, $G(a; k) = \sum_{n=0}^{\infty} \oplus (V^{\otimes n}/V^{\otimes n} \cdot Q_n)$.

We now summarize concepts and results to which we give (hook) generalizations in the sequel. The universal algebra $G(A; k)$ is used to define the Poincaré series in k (commuting) variables [4], $P(r_1, \ldots, r_k)$. Over $gl(k)$, one can write in a unique way $V^{\otimes n}/V^{\otimes n} \cdot Q_n \cong \sum_{\lambda \in H(k,0;n)} (M_\lambda^{(k)})^{\bar{m}_\lambda}$, where $M_\lambda^{(k)}$ is the irreducible $gl(k)$ module on the partition λ. It is proved in [2] that if $\chi_n(A) = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda [\lambda]$ and if $\lambda \in H(k, 0; n)$, then $\bar{m}_\lambda = m_\lambda$. Finally, it follows by results of [2, 5], by taking traces of diagonal matrices on $V^{\otimes n}/V^{\otimes n} \cdot Q_n$ that (S_λ's the Schur functions) $p(r_1, \ldots, r_k) = \sum_{n=0}^{\infty} \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda \cdot S_\lambda(r_1, \ldots, r_k)$. Note that if $h(\lambda) \geqslant k+1$, then $S_\lambda(r_1, \ldots, r_k) = 0$, hence $p(r_1, \ldots, r_k) = \sum_{n=0}^{\infty} \sum_{\lambda \in H(k,0:n)} m_\lambda \cdot S_\lambda(r_1, \ldots, r_k)$: $p(r_1, \ldots, r_k)$ captures precisely the multiplicities m_λ in $\chi_n(A)$, where $\lambda \in H(k, 0; n)$. If no strip $H(k, 0)$ contains $\chi_n(A)$ [9], then no Poincaré series $p(r_1, \ldots, r_k)$ captures all its multiplicities. This is remedied by introducing hook Poincaré series.

Now to hook generalizations! We imitate the construction of $G(A; k)$ to capture the cocharacters which are properly inside a hook. Let $V = T \oplus U$, $\dim T = k$, $\dim U = l$. In [3, Sect]. we introduced a new (right) sign permutation action of S_n on $V^{\otimes n}$, denoted by $*$. Thus $V^{\otimes n} * Q_n$ is a subspace of $V^{\otimes n}$. In fact, it is a $pl(V)$ left submodule, hence $V^{\otimes n}/V^{\otimes n} * Q_n$ is also a $pl(V)$ module.

DEFINITION 5. Let $V = T \oplus U$ be as above.

(a) $G(A; k, l) = \sum_{n=0}^{\infty} \oplus (V^{\otimes n}/V^{\otimes n} * Q_n)$.

(b) We denote $V^{\otimes n} * e_{T_\lambda}$ of [3, 3.19] by $V^{\otimes n} * e_{T_\lambda} = M_\lambda^{(k,l)}$. For each $\lambda \in H(k, l; n)$ this is an irreducible $pl(V)$ module.

(c) $V^{\otimes n}/V^{\otimes n} * Q_n = \sum_{\lambda \in H(k,l;n)} \oplus (M_\lambda^{(k,l)})^{\bar{m}_\lambda}$ as $pl(V)$ modules.

THEOREM 6. *Let $V = T \oplus U$ be as above. Let m_λ be the multiplicities in $\chi_n(A)$ and $\bar{m}_\lambda$ as in Definition 5(c). Then $\bar{m}_\lambda = m_\lambda$ for all $\lambda \in H(k, l; n)$.*

To prove this theorem we need Lemmas 7 and 8.

LEMMA 7. *Let $M \subseteq V^{\otimes n}$ be a $pl(V)$ submodule. Then there is an $e \in FS_n$ such that $M = V^{\otimes n} * e$.*

Proof. Since $V^{\otimes n}$ is completely reducible over $pl(V)$, it may be decomposed as $V^{\otimes n} = M \oplus M'$. The projection homomorphism $\pi: V^{\otimes n} \to M$ is obviously a $pl(V)$ map, i.e., $\pi \in \mathrm{Hom}_{pl(V)}(V^{\otimes n}, V^{\otimes n})$. But, by the double centralizing property of $pl(V)$ and FS_n [3, 4.15] there is an element $e \in FS_n$ such that $\pi(v) = v * e$ for all $v \in V^{\otimes n}$. Q.E.D.

LEMMA 8. *If J_1, J_2 are left ideals of FS_n and $J_1 \cap J_2 = 0$, then $(V^{\otimes n} * J_1) \cap (V^{\otimes n} * J_2) = 0$.*

Proof. Let φ be the representation we get from the $*$ action, $\varphi: FS_n \to \mathrm{hom}(V^{\otimes n}, V^{\otimes n})$. Then by complete reducibility $FS_n = A \oplus \ker \varphi$. Replacing J_i by $J_i \cap A$, we may assume wlog that for $i = 1, 2$, $J_i = FS_n e_i$, $e_i \in A$, $e_i^2 = e_i$. Let $M = (V^{\otimes n} * J_1) \cap (V^{\otimes n} * J_2)$. M is a $pl(V)$ submodule of $V^{\otimes n}$, hence by Lemma 7, M may be written as $M = V^{\otimes n} * e_3$, $e_3 \in A$. For $i = 1, 2$, $M \subseteq V^{\otimes n} * J_i$ and since e_i acts as a right unit on $V^{\otimes n} * J_i$, $v * e_3 e_i = v * e_3$ for all $v \in V^{\otimes n}$. Thus $v * (e_3 e_i - e_3) = 0$ and $e_3 e_i - e_3 \in A \cap \ker \varphi = 0$, and $e_3 e_i = e_3$, $i = 1, 2$. Finally, $e_3 \in J_1 \cap J_2 = 0$.
Q.E.D.

Proof of Theorem 6. As a left FS_n-module, $V_n = Q_n \oplus \sum_{\lambda \in \mathrm{Par}(n)} \oplus (FS_n e_\lambda)^{m_\lambda}$. By Lemma 8, $V^{\otimes n} = V^{\otimes n} * FS_n = (V^{\otimes n} * Q_n) \oplus \sum_{\lambda \in H(k,l;n)} \oplus (V^{\otimes n} * e_\lambda)^{m_\lambda}$, hence as left $pl(V)$ modules $V^{\otimes n}/V^{\otimes n} * Q_n \cong \sum_{\lambda \in H(k,l;n)} \oplus (V^{\otimes n} * e_\lambda)^{m_\lambda} \cong \sum_{\lambda \in H(k,l;n)} \oplus (M_\lambda^{(k,l)})^{m_\lambda}$. Q.E.D.

Remark 9. As a corollary, if k, l are as in Theorem 1, then $pl(V)$ decomposition of $G(a; k, l)$ into irreducible modules captures the (multiplicities of the) cocharacters of $\chi_n(A)$. We next introduce a hook version of the Poincaré series, designed to campture the cocharacters of P.I. algebras which fail to lie in any strip $H(k, 0)$.

DEFINITIONS 10. (a) Let $V = T \oplus U$ as before, $t_1,\dots, t_k \in T$, $u_1,\dots, u_l \in U$ bases inducing an identification $T(V) \equiv F\langle t_1,\dots, t_k, u_1,\dots, u_l\rangle$. Let $\langle a; b\rangle = \langle a_1,\dots, a_k; b_1,\dots, b_l\rangle$, $a_1 + \cdots + b_l = n$, and set $V\langle a; b\rangle \subseteq V^{\otimes n}$ to be the subspace of all polynomials homogeneous in each of $t_1,\dots, u_l$, of degree a_i in t_i and b_j in u_j.

(b) Let $Q \subseteq F\langle x\rangle$ be the identities of a P.I. algebra A. Define $q(r_1,\dots, r_k; s_1,\dots, s_l) = q_A(r; s) = \sum_{\langle a;b\rangle} (\dim(V\langle a; b\rangle / V\langle a; b\rangle * Q_n))\, r_1^{a_1} \cdots r_k^{a_k} s_1^{b_1} \cdots s_l^{b_l}$. We call $q(r_1,\dots, r_k; s_1,\dots, s_l)$ a double Poincaré series for A.

Recall [3, Sect. 6] that the hook Schur functions $HS_\lambda(r_1,\dots, r_k; s_1,\dots, s_l)$, $\lambda \in H(k, l; n)$ are defined by: $HS_\lambda(r_1,\dots, r_k; s_1,\dots, s_l)$ is the trace of

$$\begin{pmatrix} r_1 & & & & & 0 \\ & \ddots & & & & \\ & & r_k & & & \\ & & & s_1 & & \\ & & & & \ddots & \\ 0 & & & & & s_l \end{pmatrix}^{\otimes n} \in \operatorname{End}(V^{\otimes n}) \text{ restricted to } M_\lambda^{(k,l)} \subseteq V^{\otimes n}.$$

THEOREM 11. *For any* $k, l \geqslant 0$, $q(r_1,\dots, r_k; s_1,\dots, s_l) = \sum_{\lambda \in H(k,l)} m_\lambda HS_\lambda(r_1,\dots, r_k; s_1,\dots, s_l)$.

Proof. The left action of

$$\begin{pmatrix} r_1 & & & & & 0 \\ & \ddots & & & & \\ & & r_k & & & \\ & & & s_1 & & \\ & & & & \ddots & \\ 0 & & & & & s_l \end{pmatrix}$$

on $V^{\otimes n}$ commutes with the right FS_n action $*$ [3, Sects. 4, 6]. Hence

$$\begin{pmatrix} r_1 & & & & & 0 \\ & \ddots & & & & \\ & & r_k & & & \\ & & & s_1 & & \\ & & & & \ddots & \\ 0 & & & & & s_l \end{pmatrix}^{\otimes n}$$

acts on $(V^{\otimes n}/V^{\otimes n} * Q_n)$ and we compute the trace in two ways. First, let $x \in V^{\otimes n}$ be a monomial (basis element) of degree a_i in t_i and b_j in u_j. Then w is an eigenvector for

$$\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & s_l \end{pmatrix}$$

with eigenvalue $r_1^{a_1} \cdots r_k^{a_k} s_1^{b_1} \cdots s_l^{b_l}$. Therefore the trace of

$$\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & s_l \end{pmatrix}^{\otimes n}$$

on $(V^{\otimes n}/v^{\otimes n} * Q_n)$ is $\sum_{a_1 + \cdots + b_l = n} (\dim(V\langle a; b\rangle / V\langle a; b\rangle * Q_n)) \, r_1^{a_1} \cdots s_l^{b_l}$. On the other hand, using the decomposition $V^{\otimes n}/V^{\otimes n} * Q_n \cong \sum_{\lambda \in H(k,l;n)} \oplus (M_\lambda^{(k,l)})^{m_\lambda}$ and the definition of the HS_λ, the trace of

$$\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & s_l \end{pmatrix}^{\otimes n}$$

also equals $\sum_{\lambda \in H(k,l;n)} m_\lambda HS_\lambda(r_1,..., r_k; s_1,..., s_l)$. Q.E.D.

Note that for k, l chosen as in Theorem 1, the corresponding double Poincaré series $q_A(r_1,..., r_k; s_1,..., s_l)$ captures the cocharacter of A.

EXAMPLE 12. Let E be the infinite Grassmann algebra. By [7], $\chi_n(E) = \sum_{\lambda \in H(1,1;n)} [\lambda]$. By considering the outer products of $(1^i) \in \mathrm{Par}(i)$ and $(a) \in \mathrm{Par}(a)$, we obtain

$$S_{(1^i)}(r_1,..., r_k) \, S_{(a)}(r_1,..., r_k) = S_{(a, 1^i)}(r_1,..., r_k) + S_{(a+1, 1^{i-1})}(r_1,..., r_k)$$

and by summing, $0 \leqslant i \leqslant k$, $0 \leqslant a$, it can be shown that

$$\begin{aligned} p(r_1,..., r_k) &= p_E(r_1,..., r_k) \\ &= \frac{1 + \psi_2(r_1,..., r_k) + \psi_4(r_1,..., r_k) + \cdots + \psi_{2[k/2]}(r_1,..., r_k)}{(1 - r_1) \cdots (1 - r_k)}, \end{aligned}$$

where $\{\psi_j(r_1,..., r_k)\}_{0 \leqslant j \leqslant k}$ are the elementary symmetric polynomials. Only the infinite sequence $\{p_E(r_1,..., r_k)\}_{k=1}^{\infty}$ captures $\chi_n(E)$.

For the double Poincaré series, choose $k = l = 1$. Since $HS_{(a, 1^b)}(r; s) = r^{a-1} s^b (r + s)$ if $a \geqslant 1$ and $HS_\phi(r; s) = 1$, it follows easily that $q_E(r; s) = ((1 + rs)/(1 - r)(1 - s))$.

3

In this section we prove that the m_λ's in $\chi_n(A)$ are polynomially bounded.

LEMMA 13. *For any* P.I. *algebra A, the space $G(A: k, l)$, defined in* 5(a), *is an algebra in a natural way.*

Proof. By homogeneity,

$$G(A; k, l) = \sum_n \oplus (V^{\otimes n}/V^{\otimes n} * Q_n) = T(V)\Big/\sum_n \oplus (V^{\otimes n} * Q_n)$$

and the proof follows once we show that $\sum_n \oplus (V^{\otimes n} * Q_n) = I$ is a two-sided ideal in $T(V)$. Let $w_n \in V^{\otimes n}$, $w_m \in V^{\otimes n}$, $a_n \in Q_n$. Then it is enough to show that

$$(w_n * a_n) \otimes w_m, \qquad w_m \otimes (w_n * a_n) \in V^{\otimes n+m} * Q_{n+m}.$$

For example, $(w_n * a_n) \otimes w_m = (w_n \otimes {}_m) * \bar{a}_n$, where $a_n \to \bar{a}_n$ by the embedding $FS_n \hookrightarrow FS_{n+m}$ induced by $S_n \to S_{n+m}$ (the last m numbers are fixed).

As multilinear polynomials,

$$\bar{a}_n(x_1,..., x_{n+m}) = a_n(x_1,..., x_n)\, x_{n+1} \cdots x_{n+m};$$

hence clearly $\bar{a}_n \in Q_{n+m}$. Q.E.D.

THEOREM 14. *$G(A; k, l)$ satisfies all the identities of $A \otimes E$; in particular, $G(A; k, l)$ is* P.I. (*it might satisfy more identities than $A \otimes E$*).

Proof. Let the functions $f_I(\sigma)$ be as in [3, 6]. We first observe that $g(x_1,..., x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ is an identity for $A \otimes E$ if and only if for every $I \subseteq \{1,..., n\}$, the polynomial $\sum_{\sigma \in S_n} \alpha_\sigma f_I(\sigma)\, x_{\sigma(1)} \cdots X_{\sigma(n)}$ is an identity for A (see, for example, the proof of [6, Lemma 2.1]).

Let h be an identity for $A \otimes E$. By multilinearization, we may assume that h is multilinear, $h(x_1,..., x_n) = \sum \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$. We need to show that for every substitution

$$x_i \to d_i(t_1,..., t_k, u_1,..., u_l) \in T(V), \qquad h(d_1,..., d_n) \in \sum \oplus (V^{\otimes n} * Q_n).$$

By the multilinearity of h, it is enough to show this for $x_i \to M_i(t_1,..., t_k, u_1,..., u_l) = M_i(t, u)$, monomials.

Let the total degree of $\prod_{i=1}^{n} M_i$ be m. Factor $x_1 \cdots x_m = N_1 \cdots N_n$ such that $N_j \to M_j$ by properly substituting x's by t's and u's. Note that if $M_j = 1$.

we can choose $N_j = 1$. Clearly, for each $\sigma \in S_n$ there exists $\tau = \tau(\sigma) \in S_m$ such that $N_{\sigma(1)} \cdots N_{\sigma(n)} = (N_1 \cdots N_n)\,\tau$. Thus

$$\begin{aligned} h(N_1,..., N_n) &= \sum_{\sigma \in S_n} \alpha_\sigma N_{\sigma(1)} \cdots N_{\sigma(n)} \\ &= \sum_{\sigma \in S_n} \alpha_\sigma (N_1 \cdots N_n)\,\tau(\sigma) \\ &= N_1 \cdots N_n \cdot \sum_{\sigma \in S_n} \alpha_\sigma \tau(\sigma) \equiv \sum \alpha_\sigma \tau(\sigma); \end{aligned}$$

and clearly

$$h(M_1,..., M_n) = (M_1 \cdots M_n) \sum_\sigma \alpha_\sigma \tau(\sigma).$$

Let $M(t, u) = M_1(t, u) \cdots M_n(t, u)$ and let $J = \operatorname{supp}_U M(t, u) \subseteq \{1,..., m\}$, [3, Sect. 1], then by definition

$$M(t, u) * \tau(\sigma) = f_J(\tau(\sigma))\, M(t, u)\, \tau(\sigma),$$

so

$$M(t, u) \sum_\sigma \alpha_\sigma \tau(\sigma) = M(t, u) * \sum_\sigma \alpha_\sigma f_J(\tau(\sigma))\, \tau(\sigma).$$

Since $\sum_\sigma \alpha_\sigma \tau(\sigma)$ is a P.I. for $A \otimes E$, hence (for any J), $\sum_\sigma \alpha_\sigma f_J(\tau(\sigma))\, \tau(\sigma)$ is a P.I. for A, by the remark at the beginning of the proof $\sum_\sigma \alpha_\sigma f_J(\tau(\sigma))\, \tau(\sigma) \in Q_n$. Thus

$$h(M_1,..., M_n) = M(t, u) * \sum_\sigma \alpha_\sigma f_I(\tau(\sigma))\, \tau(\sigma) \in V^{\otimes n} * Q_n. \qquad \text{Q.E.D.}$$

COROLLARY 15. *Let A be* P.I., *$k, l \geqslant 0$ any numbers. Then the algebra $G(A; k, l)$ is finitely generated and* P.I., *hence is polynomially bounded. The sequence $a_n = \dim(V^{\otimes n}/V^{\otimes n} * Q_n)$ is polynomially bounded* [2, 4.13].

We can now prove our main result.

THEOREM 16. *Let A be any* P.I. *algebra with multilinear cocharacter $\chi_n(A) = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda[\lambda]$. Then there exists a polynomial $g_A(x) = g(x)$ such that for all n, $\sum_{\lambda \in \mathrm{Par}(n)} m_\lambda \leqslant g(n)$. In particular, the multiplicities m_λ are (uniformly) polynomially bounded.*

Note. The theorem was proved ([2]) under the assumption that A satisfies a Capelli identity. Here we prove it under no such restrictions.

Proof. Apply Theorem 1 to obtain $k, l \geqslant 0$ such that for all n, $\chi_n(A) = \sum_{\lambda \in H(k,l;n)} m_\lambda [\lambda]$, i.e., $\chi(A) \subseteq H(k,l)$. Construct $V = T \oplus U$, $\dim T = k$, $\dim U = l$, let S_n act on $V^{\otimes n}$ by the $*$ (sign permutation) action, and consider $V^{\otimes n}/V^{\otimes n} * Q_n$ as a $pl(V)$ module. By Theorem 6, $V^{\otimes n}/V^{\otimes n} * Q_n \cong_{pl(V)} \sum \oplus (M_\lambda^{(k,l)})^{m_\lambda}$, hence $\dim V^{\otimes n}/V^{\otimes n} * Q_n = \sum_{\lambda \in H(k,l;n)} m_\lambda \dim(M_\lambda^{(k,l)})$. By [3, 3.20], $M_\lambda \neq 0$ if $\lambda \in H(k, l; n)$. Thus $\dim V^{\otimes n}/V^{\otimes n} * Q_n \geqslant \sum_{\lambda \in H(k,l;n)} m_\lambda$ and the proof now follows from Corollary 15. Q.E.D.

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